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ON SINGULARITIES OF THE STRESS FUNCTION AT THE CORNER POINTS OF THE TRANSVERSE CROSS-SECTION OF A TWISTED BAR WITH A THIN REINFORCING LAYER*

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The study of twisting of prismatic bars with a thin reinforcing layer is reduced to the solution of the third boundary-value problem for Poisson's equation in the cross-sectional region of the bar /1/. However, the solution obtained by analytic or numerical methods does not always allow us to determine the singularities of the stresses near the corner points of the profile of the bar. The present article is devoted to a study of the asymptotic behavior of the solution of the third boundary-value problem for a Laplacian operator in a region with a corner point on the contour, and on this basis, the direct determination of the asymptotic properties of the stresses and coefficients of the stress intensity at the corner points of a transverse cross-section of a torsion bar with a thin reinforcing layer.

1. Asymptotic properties of the solutions of the third boundary-value problem for the Laplacian operator near the corner point of the contour.

1°. Statement of problem. Suppose Ω is a subregion of R^2 with a compact closure $\bar{\Omega}$ and boundary $\partial\Omega$ smooth everywhere other than at the origin O. We suppose that in a unit neighborhood about the point O, the region Ω coincides with the angle $K_\alpha = \{x = (x_1, x_2) \in R^2 : r \in (0, +\infty), \theta \in (0, \alpha)\}$, where (r, θ) are polar coordinates and $(0, 2\pi] \ni \alpha$ is the opening of the angle. In Ω we consider the boundary-value problem

$$\Delta u(x) = f(x), \quad x \in \Omega; \quad u(x) + \beta(x)\frac{\partial u}{\partial n}(x) = \varphi(x), \quad x \in \partial\Omega \setminus 0 \tag{1.1}$$

where n is an outer normal and β a positive function from $C^\infty(\bar{\Omega} \setminus O)$ that admits of the representation

$$\beta(x) = \sum_{k=0}^{m} r^{1+\nu_k}\beta_k(\theta) + O(r^{1+\nu_{m+1}}), \quad r \to 0 \tag{1.2}$$

Here $\{\nu_k\}$ is a strictly increasing sequence of real numbers; $\beta_k \in C^\infty([0, \alpha])$; and m is an arbitrary natural number. We further assume that (1.2) may be termwise differentiated.

In the case $\nu_0 \leqslant 0$, the conditions for the normal solvability of the problem (1.1) and the asymptotic behavior of its solutions follow from general results /2–5/ on elliptical boundary-value problems in regions with conical points. However, if $\nu_0 > 0$ and, consequently, $\beta(x) = o(r)$, Δ and $1|_{\partial K\alpha}$ are the principal (as $r \to 0$) parts of the differential operators occurring in equations (1.1). Thus in the "limiting" $(r = 0)$ problem the principal (with respect to the differential properties) part of the boundary conditions vanishes. Later in this point we will establish conditions for the normal solvability of the boundary-value problem (1.1) when the boundary conditions at the corner point degenerate, and will also find asymptotic expansions of its solutions. Note that the case in which the limiting equation in K_α is of decreasing order for general elliptical boundary-value problems in cones has been previously studied in /6–7/.

2°. Functional spaces. As in /2,4/, we let $V_\sigma^s(\Omega)$ and $V_\sigma^{s+1/2}(\partial\Omega)$ denote spaces of functions with norms

$$\|v; V_\sigma^s(\Omega)\| = \sum_{j=0}^{s} \|r^{\sigma+j-s}v; W_2^j(\Omega)\|, \; \|w; V_\sigma^{s+1/2}(\partial\Omega)\| =$$
$$\sum_{j=0}^{s} \|r^{\sigma+j-s-1/2}w; W_2^j(\partial\Omega)\| + \|r^\sigma w; W_2^{s+1/2}(\partial\Omega)\|, \quad s = 0, 1, 2, \ldots$$

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Further, suppose that $V^{s+1}_{\sigma,\delta}(\Omega)$ is a space of functions in Ω, for which the expression

$$\| u;\ V^{s+1}_{\sigma,\delta}(\Omega) \| = \| u;\ V^{s}_{\sigma-1}(\Omega) \| + \| r^{\sigma-\delta} u;\ W^{s+1}_2(\Omega) \|,\ \delta > 0$$

is finite.

3°. **Problem with a small parameter.** Suppose that $R^2 \supset \omega$ is a region with smooth boundary $\partial\omega$ (in the class C^∞) and compact closure. We consider the boundary-value problem

$$\Delta U(\varepsilon, x) = F(\varepsilon, x),\quad x \in \omega; \qquad (1.3)$$
$$U(\varepsilon, x) + \varepsilon B(x) \frac{\partial U}{\partial n}(\varepsilon, x) = \Phi(\varepsilon, x),\quad x \in \partial\omega$$

where $C^\infty(\partial\omega) \ni B$ is a positive function and ε a small parameter.

To study the boundary-value problem (1.3), we use, as is usually done, the Schwartz method of "freezing" the coefficients. Without dwelling on the usual details, we will simply state the results.

We let $H_\varepsilon^s(\omega)$ denote the S.L. Sobolev space $W_2^s(\omega)$ provided with the equivalent norm

$$\| U;\ H_\varepsilon^s(\omega) \| = \| U;\ W_2^{s-1}(\omega) \| + \varepsilon \| U;\ W_2^s(\omega) \|$$

We further introduce the sets

$$M_k^p = \{x \in K_\alpha : 2^{-k-p-1} < r < 2^{-k+p}\},\ (k = 1, 2, \ldots;\ p = 0, 1)$$

Lemma 1. If $l = 0, 1, 2, \ldots$, the mapping

$$\left\{\Delta;\ \left(1 + \varepsilon\beta(x) \frac{\partial}{\partial n}\Big|_{\partial\omega}\right\}:\ H_\varepsilon^{l+2}(\omega) \to H_\varepsilon^{l}(\omega) \times W_2^{l+1/2}(\partial\omega)$$

is an isomorphism and uniformly continuous with respect to ε.

Corollary 1. Suppose the functions U, F and Φ belong to the spaces $W_2^{l+2}(M_0^1)$, $W_2^l(M_0^1)$ and $W_2^{l+1/2}(\partial M_0^1 \cap \partial K_\alpha)$, respectively, and satisfy the equations

$$\Delta U(\varepsilon, x) = F(\varepsilon, x),\ x \in M_0^1;\ U(\varepsilon, x) + \varepsilon B(x) \frac{\partial U}{\partial n}(\varepsilon, x) =$$
$$\Phi(\varepsilon, x),\ x \in \partial M_0^1 \cap \partial K_\alpha$$

Then, the inequality is valid

$$\| U;\ H_\varepsilon^{l+2}(M_0^0) \| \leqslant c\{\| F;\ H_\varepsilon^l(M_0^1) \| + \| \Phi;\ W_2^{l+1/2}(M_0^1 \cap$$
$$\partial K_\alpha) \| + \| U;\ L_2(M_0^1) \|\}$$

and in which the constant c is independent of ε, U, F and Φ.

4°. **Auxiliary assertions.**

Lemma 2. Suppose $u \in W^{l+2}_{2,\mathrm{loc}}(\bar\Omega) \cap V^0_{\sigma-l-2}(\Omega)$ is a solution of the problem (1.1) with right sides $f \in V^l_{\sigma,\nu_0}(\Omega)$ and $\varphi \in V_\sigma^{l+1/2}(\partial\Omega)$. Then, exist the inequality

$$\| u;\ V^{l+2}_{\sigma,\nu_0}(\Omega) \| \leqslant c\{\| f; V^l_{\sigma,\nu_0}(\Omega) \| + \| \varphi; V_\sigma^{l+1/2}(\partial\Omega) \| + \| u; V^0_{\sigma-l-2}(\Omega) \| \qquad (1.4)$$

and where the constant c is independent of u.

Proof. By virtue of well-known (cf., i.e. /8/) properties of boundary-value problems in bounded domains, we need only find an estimate for the quantitiy $\| u;\ V^{l+2}_{\sigma,\nu_0}(\{x \in K_\alpha : r \leqslant 1/2\}) \|$ expressed in turns of the right side of (1.4). Moreover, we may limit ourselves to the case $\beta(x) = r^{1+\nu_0}\beta_0(\theta)$.

Following the substitution $x \to y = 2^k x$, the sets M_k^p turn into M_0^p, and the functions $v_k(y) = u(2^{-k}y)$ satisfy the equations

$$\Delta_y v_k(y) = 2^{-2k} f(2^{-k}y),\ y \in M_0^1;\ v_k(y) - 2^{-k\nu_0}\beta(\theta)\,|y|^{1+\nu_0} \frac{\partial v_k}{\partial n}(y) = \varphi(2^{-k}y),\ y \in M_0^1 \cap \partial K_\alpha$$

Together with Corollary 1, we thus obtain the estimate

$$\| v_k;\ H^{l+2}_{2^{-k\nu_0}}(M_0^0) \| \leqslant c\{2^{-2k} \| f(2^{-k}\cdot);\ H^l_{2^{-k\nu_0}}(M_0^1) \| + \qquad (1.5)$$
$$\| \varphi(2^{-k}\cdot);\ W_2^{l+1/2}(M_0^1) \| + \| v_k;\ L_2(M_0^1) \|\}$$

Multiplying relations (1.5) by $2^{k(1+l-\sigma)}$ and returning to the use of the coordinates x, using the inequalities $2^{-k} > r$ in M_k^0 and $2^{-k} < 4r$ in M_k^1, and summing the resulting estimates with respect to $k = 1, 2, 3, \ldots$, we find that

$$\| u; V_\sigma^{l+1/2}\left(\left\{x \in K_\alpha: r < \frac{1}{2}\right\}\right)\| \leqslant c \left(\| f; V^l_{\sigma,\nu_0}(\{x \in K_\alpha: r < 1\})\| + \| \varphi; V_\sigma^{l+1/2}(\{x \in \partial K_\alpha: r < 1\})\| + \| u; V^0_{\sigma-l-2}(\{x \in K_\alpha: r < 1\})\|\right)$$

The lemma is proved.

Lemma 3. Suppose $V^{l+2}_{\sigma,\nu_0}(\Omega) \ni u$ is a solution of the problem (1.1), and let $f \in V^l_{\sigma-\delta,\nu_0}(\Omega)$, $\varphi \in V^{l+1/2}_{\sigma-\delta}(\Omega)$; $\min\{\nu_0, \pi/\alpha\} > \delta > 0$; $l \geqslant 1$; the numbers $l+1-\sigma$ and $l+1-\sigma+\delta$ are not multiples of π/α. Then a) if the segment $(l+1-\sigma, l+1-\sigma+\delta)$ does not contain any numbers of the form $p\pi/\alpha$ (p an integer), then $u \in V^{l+2}_{\sigma-\delta,\nu_0}(\Omega)$; b) if $p\pi/\alpha \in (l+1-\sigma, l+1-\sigma+\delta)$, then $u_1 = u - Cr^{p\pi/\alpha}\sin(\pi p\theta/\alpha) \in V^{l+2}_{\sigma-\delta,\nu_0}(\Omega)$, where C is some constant.

Proof. By the definitions of the spaces, we have

$$u \in V^{l+1}_{\sigma-1}(\Omega), \quad f \in V^{l-1}_{\sigma-1-\delta}(\Omega), \quad \varphi \in V^{l-1/2}_{\sigma-1-\delta}(\partial\Omega)$$

Therefore, by /2/, in case a) $u \in V^{l+1}_{\sigma-1-\delta}(\Omega)$, and in case b), $u_1 \in V^{l+1}_{\sigma-1-\delta}(\Omega)$. We need only use the estimate (1.4) for the functions u or u_1.

Lemma 4. (cf. Lemma 3.1 from /4/). Let $f(x) = r^{\gamma-2}\Psi(\theta, \ln r)$, $\varphi(x) = r^\gamma\Phi(\theta, \ln r)$, when $\Psi(\theta, t)$, $\Phi(\theta, t)$ are polynomials of degree k with coefficients in the space $C^\infty([0, \alpha])$. Then a polynomial $\Xi(\theta, t)$ with coefficients in $C^\infty([0, \alpha])$ may be found such that

$$\left\{\Delta(r^\gamma\Xi(\theta, \ln r)) - r^{\gamma-2}\Phi(\theta, \ln r), \left(1 + \beta(x)\frac{\partial}{\partial n}\right)(r^\gamma\Xi(\theta, \ln r)) - r^\gamma\Psi(\theta, \ln r)\right\} \in V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_\sigma(\partial\Omega)$$

where $\sigma < l - \gamma - \nu_0$. The degree of the polynomial Ξ is equal to k if $\gamma \neq \pi p/\alpha$ and $k+1$ if $\gamma = \pi p/\alpha$.

5°. Normal solvability of boundary-value problem.

Theorem 1. The boundary-value problem (1.1) with operator

$$A = \left\{\Delta, \left(1 + \beta(x)\frac{\partial}{\partial n}\right)\right\}: V^{l+2}_{\sigma,\nu_0}(\Omega) \to V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_\sigma(\partial\Omega) \tag{1.6}$$

is normally solvable if and only if the number $l+1-\sigma$ is not a multiple of π/α (i.e., $l+1-\sigma \neq p\pi/\alpha$ for any integer p).

Proof. Finite-dimensionality of the kernel. If $u \in V^{l+2}_{\sigma,\nu_0}(\Omega)$ is a solution of the homogeneous problem (1.1), by virtue of Lemmas 2 and 3, we may find a number $\delta > 0$ such that $u \in V^{l+3}_{\sigma+1-\delta,\nu_0}(\Omega)$. Since the embedding $V^{l+2}_{\sigma,\nu_0}(\Omega) \subset V^{l+3}_{\sigma+1-\delta,\nu_0}(\Omega)$ is compact, then $\dim\ker A < +\infty$.

Left regularizer. We construct the operator

$$R: V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_\sigma(\partial\Omega) \to V^{l+2}_{\sigma,\nu_0}(\Omega) \tag{1.7}$$

such that the mapping

$$1 - AR: V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_\sigma(\partial\Omega) \to V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_\sigma(\partial\Omega) \tag{1.8}$$

is completely continuous.

From Lemma 1 and Corollary 1 it follows that there exist operators

$$P(k): H^l_{2^{-k}\nu_0}(M_0^1) \times W_2^{l+1/2}(\partial M_0^1 \cap \partial K_\alpha) \to H^{l+2}_{2^{-k}\nu_0}(M_0^1) \tag{1.9}$$

such that the mappings

$$\chi(\ln|y|)1 - \left\{\Delta, (1 + \beta(2^{-k}y)\frac{\partial}{\partial n})_{|\partial M_0^1 \cap \partial K_\alpha}\right\}\chi(\ln|y|)P(k): H^l_{2^{-k}\nu_0}(M_0^1) \times W_2^{l+1/2}(\partial M_0^1 \cap \partial K_\alpha) \to H^{l+1}_{2^{-k}\nu_0}(M_0^1) \times W_2^{l+3/2}(\partial M_0^1 \cap \partial K_\alpha) \tag{1.10}$$

are continuous, further the norms of the operators (1.9) and (1.10) are bounded by constant and independent of k. Here $\chi \in C^\infty((-1, 2))$; $\chi(t) = 1$ if $t \in [0, 1]$; $\chi(t) + \chi(t+2) = 1$ if $t \in [0, 3]$. We introduce the operator

$$T(f,\varphi)(x) = \sum_{k=2}^{\infty} \chi(2k - \ln r)\{P(2k)(f,\varphi)(2^{-2k}y)\}|_{x=2^{-2k}/y} +$$
$$\zeta\left(\ln\frac{1}{r}\right) L(f,\varphi)(x);\ \zeta(t) = 1 - \sum_{k=2}^{\infty} \chi(t+2k)$$

where L is the left regularizer in problem (1.1) and in $\Omega \cap \{x \in R^3 : r \leqslant 2\}$ /8/. We may verify that the mapping

$$T : V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_{\sigma}(\partial\Omega) \to V^{l+2}_{\sigma,\nu_0}(\Omega) \tag{1.11}$$

is continuous and, in addition, the operator

$$(1 - AT)^3 : V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_{\sigma}(\partial\Omega) \to V^{l+3}_{\sigma-3,\nu_0}(\Omega) \times V^{l+7/2}_{\sigma-3}(\partial\Omega) \tag{1.12}$$

is also bounded.

We define R by the formula

$$R = T + T(1 - AT) + T(1 - AT)^2 + \left(1 - \zeta\left(\ln\frac{1}{r}\right)\right) D^{-1}(1 - AT)^3$$

where D^{-1} is the operator inverse to the operator

$$\{\Delta, 1|_{\partial K_\alpha}\} : V^{l+4}_{\sigma-2}(\Omega) \to V^{l+2}_{\sigma-2}(\Omega) \times V^{l+5/2}_{\sigma-2}(\partial\Omega) \tag{1.13}$$

(which exists by virtue of previous results /2/ and the above conditions). Clearly, the mapping (1.7) is continuous. Moreover, by (1.11)–(1.13) the operator

$$1 - AR = -\left\{\left[\Delta, \zeta\left(\ln\frac{1}{r}\right)\right],\ \left(1 - \zeta\left(\ln\frac{1}{r}\right)\right)\beta(x)\frac{\partial}{\partial n}\Big|_{\partial\Omega}\right\}(1 - AT)^3$$

acts in the space

$$V^{l+1}_{\sigma-1-\varkappa,\nu_0}(\Omega) \times V^{l+3/2}_{\sigma-1-\varkappa}(\partial\Omega),\ \varkappa = \min\{\nu_0, \nu_1 - \nu_0\}$$

which is embedded compactly in $V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_{\sigma}(\partial\Omega)$. Therefore the mapping (1.8) is completely continuous.

Closure of subspace $\operatorname{Im} A$. We need only verify that the following inequality is valid:

$$\| u; V^{l+2}_{\sigma,\nu_0}(\Omega) \| \leqslant c \| Au; V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_{\sigma}(\partial\Omega) \| \tag{1.14}$$

for any function $u \in S$, where S is the direct complement of $\ker A$ with respect to $V^{l+2}_{\sigma,\nu_0}(\Omega)$. We assume that the estimate (1.14) is false. Then a sequence $\{u_j\}$ of elements of the space $V^{l+2}_{\sigma,\nu_0}(\Omega)$ may be found such that $\| u; V^{l+2}_{\sigma,\nu_0}(\Omega) \| = 1$, while the right side of inequality (1.14) tends to zero as $j \to \infty$. We find a subsequence that converges weakly in $V^{l+2}_{\sigma,\nu_0}(\Omega)$ to some function u. Without any limitation on generality, we may assume that the sequence $\{u_j\}$ itself possesses this property. Since $\| Au_j; V^l_{\sigma,\nu_0}(\Omega) \times V^{l+1/2}_{\sigma}(\partial\Omega) \| \to 0$, then $Au = 0$. Setting $v_j = -RA\,u_j$, we have $\| v_j; V^{l+2}_{\sigma,\nu_0}(\Omega) \| \to 0$. From the properties of the operator proved above, we obtain the inclusion

$$A(u_j + v_j) \in V^l_{\sigma-\delta,\nu_0}(\Omega) \times V^{l+1/2}_{\sigma-\delta}(\partial\Omega)$$

where $\delta > 0$. Consequently, we have valid the chain of inequalities

$$\begin{aligned} \| u_j + v_j - u;\ & V^{l+2}_{\sigma-\delta,\nu_0}(\Omega) \| \leqslant c\{\| u_j + v_j - u;\ V^{l+2}_{\sigma,\nu_0}(\Omega) \| + \\ & \| A(u_j + v_j);\ V^l_{\sigma-\delta,\nu_0}(\Omega) \times V^{l+1/2}_{\sigma-\delta}(\partial\Omega) \|\} \leqslant c\{\| u_j + v_j - u; \\ & V^0_{\sigma-l-2}(\Omega) \| + \| A(u_j + v_j);\ V^l_{\sigma-\delta,\nu_0}(\Omega) \times V^{l+1/2}_{\sigma-\delta}(\partial\Omega) \|\} \end{aligned} \tag{1.15}$$

It follows from (1.15) that, in particular, the sequence $u_j + v_j$ converges strongly in $V^{l+2}_{\sigma,\nu_0}(\Omega)$ to u. But then $u \in \ker A$, and we have obtained a contradiction.

The three properties of the boundary-value problem (1.1) we have proved demonstrate that the normal solvability conditions of the theorem stated therein are sufficient. That they are necessary may be established precisely as in /2/. The theorem is proved.

Note that this technique of proving normal solvability may be carried over without any changes to the general elliptical boundary-value problem with degenerate boundary conditions at a conical point.

6°. Asymptotic solutions. We assume that the right sides of f and φ of problem (1.1) admit of the asymptotic expansions

$$f(x)=\sum_{n=1}^{N} r^{l-1-\sigma+\lambda_n} f_n(\theta, \ln r)+f_N^*(x) \tag{1.16}$$
$$\varphi(x)=\sum_{n=1}^{N} r^{l+1-\sigma+\lambda_n} \varphi_n(\theta, \ln r)+\varphi_N^*(x)$$

where $\{\lambda_n\}$ is a strictly increasing sequence of positive numbers; $f_n(\theta, t)$ and $\varphi_n(\theta, t)$ are polynomials in t with coefficients in $C^\infty([0, \alpha])$; and the residues satisfy the inclusion relations

$$f_N^* \in V^l_{\beta+\lambda_N, \nu_0}(\Omega), \quad \varphi_N^* \in V^{l+2}_{\beta+\lambda_N}(\Omega) \tag{1.17}$$

We let $\{\gamma_n\}$ denote an ordered sequence of numbers γ that may be represented in the form

$$\gamma=\lambda_n+\sum_{j=0}^{J} n_j \nu_j \quad \text{or} \quad \gamma=\frac{\pi p}{\alpha}-l-1+\sigma+\sum_{j=0}^{J} n_j \nu_j \tag{1.18}$$

where J and n_j are arbitrary non-negative integers and p is an integer that exceeds $\alpha(\sigma-1-l)/\pi$.

From Lemmas 3 and 4, it follows the theorem:

Theorem 2. The solution $u \in V^{l+2}_{\sigma,\nu_0}(\Omega)$ of problem (1.1) with right sides that admit of the expansions (1.16) and (1.17) satisfies the formula

$$u(x)=\sum_{n=0}^{N} r^{l+1-\sigma+\gamma_n} u_n(\theta, \ln r)+u_N^*(x)$$

where $u_n(\theta, t)$ are polynomials in t with coefficients in $C^\infty([0, \alpha])$; N is an arbitrary natural number; and $u_N^* \in V^{l+2}_{\beta+\gamma_N}(\Omega)$. If, in addition, $\deg f_n=\deg\varphi_n=0$ and if the number $l+1-\sigma+\pi q/\alpha$ appears among the numbers (1.18) only when $p=q$ and $n_j=0$, the degrees $\deg u_n$ of the polynomials $u_n(\theta, t)$ are also equal to zero.

7°. Unique solvability of boundary-value problem. Since $\beta(x)>0$ in (1.1) whenever $x\in\partial\Omega \setminus O$, then the problem (1.1) with homogeneous boundary condition (i.e., $\varphi\equiv 0$) $W_2^1(\Omega)$ is elliptical and, consequently, uniquely solvable in $W_2^1(\Omega)$ for any $f\in L_2(\Omega)$. By virtue of this obvious fact, the following assertion regarding the dimension of the subspaces $\ker A$ and $\operatorname{coker} A$ may be derived from Theorems 1 and 2.

Theorem 3. Suppose that $\nu_0>0$ and that the number $l+1-\sigma$ is not a multiple of π/α. Then:

a) if $\sigma>l+1-\pi/\alpha$, the cokernel of the operator (1.6) is trivial, while $\dim\ker A=[(\sigma-l-1)\alpha/\pi]$;

b) if $\sigma<l+1+\pi/\alpha$, the kernal of the operator (1.6) is trivial, and $\dim\cdot\operatorname{coker} A=[(1+l-\sigma)\alpha/\pi]$. It follows from Theorem 3 that, in particular, the problem (1.1) is uniquely solvable in the space $V^{l+2}_{\sigma,\nu_0}(\Omega)$ for any $f\in V^l_{\sigma,\nu_0}(\Omega)$ and $\varphi\in V^{l+1/2}_{\sigma}(\partial\Omega)$ if $\sigma\in(l+1-\pi/\alpha, l+1+\pi/\alpha)$, i.e., for these values of σ the mapping (1.6) is an isomorphism.

2. Features of the stress function at the corner points of the cross-section of a torsion bar with thin reinforcing coating.

1°. Statement of problem. It is known /1/ that the torsion problem for a prismatic bar reinforced by a thin layer reduces to the determination of a stress function U in the cross-sectional region Ω that satisfies the equations

$$\Delta U(x)=-2G, \; x\in\Omega \tag{2.1}$$
$$U(x)+g\delta(x)\frac{\partial U}{\partial n}(x)=0, \quad x\in\partial\Omega; \quad g=\frac{G}{G_c} \tag{2.2}$$

where G and G_c are the shear moduli for the materials of the bar and layer; $\delta(x)$ is the thickness of the layer at the point $x\in\partial\Omega$ measured along the outer normal n; and $\delta(x)$ is a small quantity. Suppose the simply connected region Ω whose characteristic dimension is scaled to unity has as boundary a contour with corner point O and angle $\alpha\in(0, 2\pi]$. We will assume that $\delta(0) = 0$. Previous results /2/ and results of Sect.1 of the present paper may be used to study the influence of the nature and degree of thinning of the layer near the corner point on the asymptotic behavior of the stress function U. More precisely, we will find the form of the asymptote U as $r \to 0$ and as a function of the parameter $\gamma \in (0, +\infty)$ which characterizes the behavior of U near the corner point

$$\delta(x) = \delta_\pm r^\gamma + O(r^{\gamma+\sigma}) \text{ for } r \to 0, x \in \Gamma_\pm$$

where $\delta_\pm$, and σ are positive constants, and Γ_- and Γ_+ denote the parts of the contour $\partial\Omega$ corresponding to the values $\theta = 0$ and $\theta = \alpha$ of the angular variable.

2°. Auxiliary assertions.

Lemma 5. The nonzero roots of the equation

$$\sin(\lambda\alpha)\,[1 - (\lambda g)^2\delta_-\delta_+] + \lambda\cos(\lambda\alpha)\,g\,(\delta_+ + \delta_-) = 0 \tag{2.3}$$

and only the nonzero roots are the eigenvalues of the spectral problem

$$\frac{\partial^2\Phi}{\partial\theta^2}(\theta) + \lambda^2\Phi(\theta) = 0, \quad \theta \in [0, \alpha]; \qquad \Phi(\theta) \pm g\,\delta_\pm \frac{\partial\Phi}{\partial\theta}(\theta) = 0, \quad \theta = \frac{\alpha}{2} \pm \frac{\alpha}{2} \tag{2.4}$$

These numbers are real, have single multiplicity, and are associated with the eigenfunctions

$$\Phi(\theta) = \sin(\lambda\theta) + \lambda g\delta_- \cos(\lambda\theta) \tag{2.5}$$

Proof. If $\lambda = 0$, the boundary-value problem (2.4) is single-valued solvable. If $\lambda \neq 0$, the linear combination $a\sin(\lambda\theta) + b\cos(\alpha\theta)$ is a solution of the equation from problem (2.4) along the segment $[0, \alpha]$. Bearing in mind the boundary conditions in (2.4), we find that this function is nonzero and satisfies problem (2.4) only if $b = \lambda g\delta_- a \neq 0$ where $\lambda \neq 0$ is a root of equation (2.3). Since the problem (2.4) is self-adjoint, all its eigenvalues are real. In addition, the same eigenfunctions (2.5) correspond to the eigenvalues λ and $-\lambda$, and, consequently, there are no adjoint vectors.

The next three assertions may be verified by direct computation.

Lemma 6. The particular solution V_1 of the boundary-value problem

$$\Delta V(x) = -2G, x \in K_\alpha; \; V(x) = 0, \; x \in \partial K_\alpha$$

at the angle $K_\alpha = \{x \in R^2 : r > 0, \theta \in (0, \alpha)\}$ has the form

$$V_1(x) = \begin{cases} Gr^2 \sin\theta \sin(\alpha - \theta)(\cos\alpha)^{-1}, & \theta \neq \pi/2, \; 3\pi/2 \\ Gr^2 \{[\sin 2\theta \ln r + \theta\cos 2\theta]\,(\alpha\cos 2\alpha)^{-1} - \sin^2\theta\}, & \\ \theta = \pi/2, \; 3\pi/2 & \end{cases} \tag{2.6}$$

Lemma 7. The particular solution V_2 of the boundary-value problem

$$\Delta V(x) = 0, \quad x \in K_\alpha; \; \frac{\partial V}{\partial\theta}(x) = \mp \frac{r^{1-\gamma}}{g\delta_-}, \quad x \in \partial K_\alpha \qquad \theta = \frac{\alpha}{2} \pm \frac{\alpha}{2}$$

is determined by means of the equation (k is an integer)

$$V_2(x) = \begin{cases} \dfrac{r^{1-\gamma}}{1-\gamma} g^{-1}\left\{\dfrac{1}{\delta_-}\sin(1-\gamma)\theta + \dfrac{\delta_-^{-1}\cos(1-\gamma)\alpha + \delta_+^{-1}}{\sin(1-\gamma)\alpha}\cos(1-\gamma)\theta\right\} \\ (1-\gamma)\alpha \neq k\pi \\ \dfrac{r^{\pi k/\alpha}}{\pi k} g^{-1}\left\{\left(\dfrac{1}{\delta_-} + \dfrac{\cos\pi k}{\delta_+}\right)\left(\ln r \cos\dfrac{\pi k}{\alpha}\theta - \theta\sin\dfrac{\pi k}{\alpha}\theta\right) + \right. \\ \left. \dfrac{\alpha}{\delta_-}\sin\dfrac{\pi k}{\alpha}\theta\right\}, \; (1-\gamma)\alpha = k\pi \end{cases} \tag{2.7}$$

Lemma 8. The particular solution V_3 of the boundary-value problem

$$\Delta V(x) = -2G, \quad x \in K_\alpha; \quad V(x) \pm g\delta_\pm \frac{\partial V}{\partial\theta}(x) = 0 \qquad x \in \partial K_\alpha, \quad \theta = \frac{\alpha}{2} \pm \frac{\alpha}{2}$$

is specified by the formulas

$$V_3(x)=\begin{cases} Gr^2\left\{a\sin 2\theta+\left(\frac{1}{2}+2\delta_-ga\right)\cos 2\theta-\frac{1}{2}\right\} \\ \lambda_j\neq 2,\ j=0,\pm 1,\dots \\ Gr^2\{\Lambda\ln r\,[\sin 2\theta+2g\delta_-\cos 2\theta]+\Psi(\theta)\},\quad \lambda_k=2 \end{cases} \tag{2.8}$$

Here λ_j are the roots of equation (2.3); the constants a and Λ have the form

$$a=\frac{1}{2}[1-\cos 2\alpha+2\delta_+g\sin 2\alpha][\sin 2\alpha(1-4g^2\delta_+\delta_-)+2g(\delta_-+\delta_+)\cos 2\alpha]^{-1}$$

$$\Lambda=-(1-\cos 2\alpha+2\delta_-g\sin 2\alpha)\left[2\alpha(1+4g^2\delta_-^2)-\frac{1}{2}\sin 4\alpha(1-4g^2\delta_-^2)+2g\delta_-(1-\cos 4\alpha)\right]^{-1}$$

where Ψ is a solution of the boundary-value problem for the ordinary differential equation

$$\frac{\partial^2\Psi}{\partial\theta^2}(\theta)+4\Psi(\theta)=-2G(2\Lambda[\sin 2\theta+2g\delta_-\cos 2\theta]+1),\quad 0\in[0,\alpha]$$

$$\Psi(\theta)\pm g\delta_\pm\frac{\partial\Psi}{\partial\theta}(\theta)=0,\quad \theta=\frac{\alpha}{2}\pm\frac{\alpha}{2}$$

3°. Asymptotic behavior of solutions at a corner point.

Depending on the parameter γ, one of the operators

$$\pm r^{\gamma-1}g\delta_\pm\frac{\partial}{\partial\theta}\Big|_{\partial K_\alpha}\quad \text{if}\quad \gamma\in(0,1) \tag{2.9}$$

$$\left(1\pm g\delta_\pm\frac{\partial}{\partial\theta}\right)\Big|_{\partial K_\alpha}\quad \text{if}\quad \gamma=1 \tag{2.10}$$

$$1|_{\partial K_\alpha}\quad \text{if}\quad \gamma>1 \tag{2.11}$$

is the principal part of the boundary conditions (2.2).

The asymptotic formulas presented below for the function U follow, for example, from /2, 3/ in case (2.9), from /1,2/ in case (2.10), and in case (2.11) are consequences of the results of Sect.1.

Let $0<\gamma<1$. Then the following representations hold:

$$U(x)=\begin{cases} C+CV_2(x)+o(r^{\chi_1}),\quad \alpha<\pi(1-\gamma)^{-1} \\ C+CV_2(x)+C_1r^{\pi/\alpha}\cos\frac{\pi\theta}{\alpha}+o(r^{\chi_2}),\quad \alpha=\pi(1-\gamma)^{-1} \\ C+C_1r^{\pi/\alpha}\cos\frac{\pi\theta}{\alpha}+o(r^{\chi_3}),\quad \alpha>\pi(1-\gamma)^{-1} \end{cases} \tag{2.12}$$

where V_2 is the function of (2.7); C and C_1 are given certain constants; and χ_j are arbitrary numbers that satisfy the inequalities

$$\chi_1<\sigma+1-\gamma,\ \chi_1<\pi/\alpha,\ \chi_2<\sigma+1-\gamma,\ \chi_2<2\pi/\alpha,$$
$$\chi_3<1-\gamma,\ \chi_3<2\pi/\alpha$$

If $\gamma=1$, then for function U satisfies the relations

$$U(x)=\begin{cases} C_2r^{\lambda_1}[\sin\lambda_1\theta+\lambda_1g\delta_-\cos\lambda_1\theta]+o(r^{\chi_1}),\quad \lambda_1<2 \\ V_3(x)+C_2r^2[\sin 2\theta+2g\delta_-\cos 2\theta]+o(r^{\chi_2}),\quad \lambda_1=2 \\ V_3(x)+o(r^{\chi_3}),\quad \lambda_1>2 \end{cases} \tag{2.13}$$

where λ_1 is the smallest positive root of equation (2.3); λ_2, the next largest positive root of (2.3); C_2, a given certain constant; V_3, the function (2.8); and χ_j, arbitrary numbers that satisfy the inequalities

$$\chi_1<2,\ \chi_1<\lambda_1+\sigma,\ \chi_1<\lambda_2,\ \chi_2<2+\sigma,\ \chi_2<\lambda_2,\ \chi_3<\lambda_1,\ \chi_3<2+\sigma$$

Finally, if $\gamma>1$, we find from Theorem 2 that

$$U(x)=\begin{cases} C_3r^{\pi/\alpha}\sin(\pi\theta/\alpha)+o(r^{\chi_1}),\quad \alpha>\pi/2 \\ V_1(x)+C_3r^2\sin 2\theta+o(r^{\chi_2}),\quad \alpha=\pi/2 \\ V_1(x)+o(r^{\chi_3}),\quad \alpha<\pi/2 \end{cases} \tag{2.14}$$

Here V_1 is the function of (2.6); C_3, is some constant; and χ_j, numbers that obey the inequalities

$$\chi_1 < 2,\ \chi_1 < 2\pi/\alpha,\ \chi_1 < \gamma - 1 + \pi/\alpha,\ \chi_2 < \gamma + 1,\ \chi_2 < 4,$$
$$\chi_3 < \gamma + 1,\ \chi_3 < \pi/\alpha$$

Let us emphasize that formulas (2.12)–(2.14) includes the case $\alpha = \pi$, which corresponds to the smooth boundary $\partial\Omega$, but also to a degeneracy of the boundary conditions at the point O, i.e., $\delta(0) = 0$.

Note an interesting fact that follows from formula (2.12). That is, if $0 < \gamma < 1$ and $\alpha < \pi$, the tangential stresses in a torsion bar with a thin reinforcing layer that are expressed in terms of the stress function U will have singularities at the corner points of the bar cross-section. At first glance, this paradoxical circumstance may be attributed to the fact that the cross-section of a torsion bar without reinforcing layer has a departure angle which, because of the reinforcing layer, becomes an entrance angle. In particular, if $\gamma < 1$, the tangential stresses have singularities at the vertex of both the entrance and departure angles of the cross-section of a torsion bar with reinforcing layer. Then, as in the case of twisting of a bar without a layer, it is well known /1/ that the stresses have singularities only at the vertex of the entrance angle of the cross-section.

4°. **Asymptotic behavior of solutions at infinity.** We present results on the asymptote of the solutions of the boundary-value problem

$$\Delta W(x) = f(x),\quad x \in \omega,\quad W(x) + \beta(x) g^{-1} \frac{\partial W}{\partial n}(x) = 0,\quad x \in \partial\omega \qquad (2.15)$$

in a region ω having an "angular exit" on infinity, i.e., at high values of r the region ω coincides with the angle K_α. The need for studying asymptotic expansions of solutions of such problems as (2.15) arises in considering torsion bars which have been strengthened by a thin reinforcing layer and which are in the form of a sector with lateral side of length D. At high values of D, the behavior of the stress function far from the vertex of the sector is described (cf. /9/) with the asymptotic representation of the solution W of problem (2.15) when $f = -2G$ and $\omega = K_\alpha$. Note, too, that the boundary-value problem (2.15) is of the same type as the problem with stationary distribution of temperature in a wedge whose sides are in spontaneous heat exchange with the environment. To simplify the formulations, we assume that the right side f in problem (2.15) is finite. The necessary particular solutions at the angle K_α for $f(x) = -2G$ are presented in Lemmas 6 and 8. We also assume that when $\theta = \alpha/2 \pm \alpha/2$ and $r > r_0 > 0$, we have $\delta(x) = \delta_\pm r^{-\nu}$ which are achieved. Since equations (2.15) are valid only for bars with a thin layer, the function δ cannot grow faster than as a linear function, consequently $\nu \geqslant -1$.

By means of conformal mapping, the region ω may be transformed into the region Ω described in Sect.1, 1°, so that the asymptotic formulas presented above are seen as justified in light of the results cited in Sect.3°. Thus, when $\nu = -1$ and $\nu > -1$, the solution W of problem (2.15) satisfies the formulas

$$W(x) = C^{(1)} r^{-\lambda_1} [\sin \lambda_1\theta + \lambda_1 g \delta_- \cos \lambda_1\theta] + O(r^{-\lambda_2}),\quad r \to \infty$$
$$W(x) = C^{(2)} r^{-\pi/\alpha} \sin \frac{\pi\theta}{\alpha} + o(r^{-\chi}),\quad r \to \infty$$

respectively. Here $C^{(1)}$ and $C^{(2)}$ are some constants; λ_1 and λ_2, the first two positive roots of equation (2.3); and $\chi < \min\{2\pi/\alpha, -1 + \nu + \pi/\alpha\}$.

5°. **Formulas for the intensity coefficients.** By means of the method of /4/, we now write out explicit representations for the coefficients C_j occurring in the asymptotic formulas (2.12)–(2.14). Since the use of the method of /4/ in problems of the theory of elasticity has been repeatedly discussed (cf. /10–12/), we will discuss only one of the possible variants. That is, we consider the case $\gamma > 1$, $\alpha > \pi/2$, (cf. the formula (2.14)).

We let ζ denote a function harmonic in $\overline{\Omega} \setminus O$ that satisfies the boundary conditions

$$\zeta(x) + \delta(x) g \frac{\partial \zeta}{\partial n}(x) = 0,\quad x \in \partial\Omega$$

and the relation

$$\zeta(x) = r^{-\pi/\alpha} \sin \frac{\pi\theta}{\alpha} + o(r^{-\pi/\alpha}),\quad r \to 0$$

There can be no doubt as to the existence of such a function. In fact,

$$\zeta(x)=\eta(x)r^{-\pi/\alpha}\sin\frac{\pi\theta}{\alpha}+z(x)$$

where η is a truncating function equal to zero outside a small neighborhood of the point O and to unity close to this point, and z is a solution of the boundary-value problem

$$\Delta z(x)=-r^{-\pi/\alpha}\sin\frac{\pi\theta}{\alpha}\Delta\eta(x)-2\,\mathrm{grad}\left(r^{-\pi/\alpha}\sin\frac{\pi\theta}{\alpha}\right)\mathrm{grad}\,\eta(x),\quad x\in\Omega \tag{2.16}$$

$$z(x)+g\delta(x)\frac{\partial z}{\partial n}(x)=-g\delta(x)\frac{\partial}{\partial n}\left[\eta(x)r^{-\pi/\alpha}\sin\frac{\pi\theta}{\alpha}\right],\quad x\in\partial\Omega$$

From Theorem 1 and the fact that the form generated by the problem (2.1), (2.2) is positive-definite, it follows that the problem (2.16) is uniquely solvable in the class of functions that obey the condition $z(x)=o(r^{-\pi/\alpha})$ as $r\to 0$.

Applying Green's formula to the functions ζ and U in the region $\Omega_d=\{x\in\Omega:\ r>d\}$ and passing, as is usual, to the limit as $d\to 0$, we find that

$$\begin{aligned}&\int_\Omega\{-2G\zeta(x)\}\,dx=\lim_{d\to0}\int_{\Omega_d}\{-2G\zeta(x)\}\,dx=\lim_{d\to0}\int_{\Omega_d}\Delta U(x)\zeta(x)\,dx=\\&\lim_{d\to0}\int_{\partial\Omega_d}\left\{\frac{\partial U}{\partial n}(x)\zeta(x)-\frac{\partial\zeta}{\partial n}(x)U(x)\right\}ds=\\&\lim_{d\to0}\int_{\partial\Omega_d\setminus\Gamma_d}\left\{\left[\frac{\partial U}{\partial n}(x)+(g\delta(x))^{-1}U(x)\right]\zeta(x)-\right.\\&\left.\left[\frac{\partial\zeta}{\partial n}(x)+g(\delta(x))^{-1}\zeta(x)\right]U(x)\right\}ds-\\&\int_{\Gamma_d}\left\{\frac{\partial U}{\partial r}(x)\zeta(x)-\frac{\partial\zeta}{\partial r}(x)U(x)\right\}ds\Big\}=\\&-\lim_{d\to0}C_3\int_0^\alpha\left\{2\frac{\pi}{\alpha}\left(\sin\frac{\pi\theta}{\alpha}\right)^2+o(1)\right\}d\theta=-\pi C_3\\&\Gamma_d=\{x\in K_\alpha: r=d\}\end{aligned} \tag{2.17}$$

Thus

$$C_3=\frac{2G}{\pi}\int_\Omega\zeta(x)\,dx$$

The function ζ is constructed by finding the solution of problem (2.16), a problem that represents considerable difficulty. On the other hand, if an exact or approximate solution of problem (2.1), (2.2) is found by some method, then its behavior near the corner point may be determined by means of the method proposed in /12/. Performing the computations as in (2.17) in the truncated sector $S_{D,d}=\{x\in K_\alpha: D>r>d\}$ for the function $Z(x)=(r^{-\pi/\alpha}-D^{-2\pi/\alpha}r^{\pi/\alpha})\sin(\pi\theta/\alpha)$ we have

$$2G\int_{S_{D,d}}Z(x)\,dx-\int_{\partial S_{D,0}}\frac{\partial Z}{\partial n}(x)U(x)\,ds=\pi C_3 \tag{2.18}$$

By means of this formula, we may find the intensity coefficient based on the values of U along the arc Γ_D and the boundary of the region $\partial\Omega$. Note that by virtue of (2.14), $U(x)|_{\partial\Omega}=o(r^{-\pi/\alpha+\sigma}),\ \sigma>0$, and the integral along $\partial S_{D,0}$ in (2.18) converges. Moreover,

$$\int_{S_{D,0}}Z(x)\,dx=\frac{4D^{2-\pi/\alpha}}{4-(\pi/\alpha)^2}$$

and, finally, formula (2.18) assumes the form

$$C_3=\frac{8GD^{2-\pi/\alpha}}{\pi(4-(\pi/\alpha)^2)}-\frac{1}{\pi}\int_{\partial S_{D,0}}\frac{\partial Z}{\partial n}(x)U(x)\,ds$$

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